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Characterizations of Boolean Algebras of Idempotents

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The notions of the left (right) Jordan groupoids are introduced. If R is an associative $*$ ring with the identity and if $U(R)$ [resp. $P(R)$] denotes the set of all idempotents (resp. projections) of the $*$ ring R, then the operations $p \underset{1}{\circ} q = p - 2pq - 2qp + 4qpq$ and $p \underset{2}{\circ} q = q - 2pq - 2qp + 4pqp$, if $p, q \in U(R)$ [resp. $p, q \in P(R)$], are the nonassociative linear operations in $U(R)$ [resp. in $P(R)$]. The present paper shows that the operations $\underset{1}{\circ}$ and $\underset{2}{\circ}$ are associative iff $pq = qp$ for $p, q \in U(R)$ [resp. $p, q \in P(R)$]. As a corollary it follows from this that the orthomodular poset $(U(R), \leq, 0, 1, ')$ is a Boolean algebra [which is commutative, i.e., $pq = qp$, $p, q \in U(R)$] iff $(U(R), \underset{1}{\circ}, 0, 1, ')$ or $(U(R), \underset{2}{\circ}, 0, 1, ')$ are Jordan associative groupoids. Similar results hold for $(P(R), \leq, 0, 1, ')$.

INTRODUCTION

Let R be an associative $*$ ring with the identity, and let $U(R)$, resp. $P(R)$, be the sets of all idempotents, resp. projectors, of the $*$ ring R. (The element $e \in R$ is an idempotent, resp. a projector, of R if $e^2 = e$, resp. $e^2 = e = e^*$.)

Easy ring-theoretic computations show that the definition

$$e \leq f \Leftrightarrow ef = fe = e, \qquad e, f \in U(R) \text{ [resp. } e, f \in P(R)]$$

yields to a partial ordering on $U(R)$, resp. $P(R)$. If the ring R has an identity, and if $e \in U(R)$, resp. $e \in = P(R)$, then the setting $e' = 1 - e$ defines the orthocomplementations on $U(R)$, resp. $P(R)$. It is well known (Flachsmeyer, 1982; Katrnoška, 1980) that the sets $U(R)$, resp. $P(R)$, are in general the orthomodular orthocomplemented posets, which need not be the lattices.

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Another way of characterizing the set $U(R)$, resp. $P(R)$, of all idempotents, resp. projectors, of the $*$ ring R gives so-called left (right) Jordan groupoids of idempotents $U(R)$, resp. projectors $P(R)$ of the $*$ ring R.

For $p, q \in U(R)$, resp. $p, q \in P(R)$, we define

$$p \underset{1}{\circ} q = p - 2pq - 2qp + 4qpq$$

$$p \underset{2}{\circ} q = q - 2pq - 2qp + 4pqp$$

It can be shown that $p \underset{1}{\circ} q$ and $p \underset{2}{\circ} q$ belongs to $U(R)$. [When $p, q \in P(R)$, then $p \underset{1}{\circ} q$ and $p \underset{2}{\circ} q$ belong to $P(R)$.] For every $p \in U(R)$ [resp. $p \in P(R)$] we put $p' = 1 - p$, and we claim that p' is an orthocomplement of p. The sets $(U(R), \underset{i}{\circ}, 0, 1, ')$ and $(P(R), \underset{i}{\circ}, 0, 1, ')$ are then the Jordan groupoids of the idempotents, resp. projectors, of the $*$ ring R.

Katrnoška (1980) shows that the elements $p, q \in U(R)$, resp. $p, q \in P(R)$, are orthogonal (we write then $p \perp q$) if $pq = qp = 0$ and $p, q \in U(R)$, resp. $p, q \in P(R)$, are compatible if $pq = qp$.

1. SOME NOTIONS AND DEFINITIONS

We can now formalize the situation in the following definition.

Definition 1.1 (Katrnoška, 1993). The nonempty set $X \neq 0$ will be called a *left Jordan groupoid* if on X are defined a binary operation $\circ$: $X \times X \to X$ and a unary operation $'$: $X \to X$ so that:

(i) $p \circ p = p$ if $p \in X$.
(ii) $(p \circ q) \circ p = p \circ (q \circ p)$, $p, q \in X$.
(iii) $(p \circ q) \circ q = p$, if $p, q \in X$.
(iv) $(p')' = p$, $p \in X$.
(v) $(p \circ q)' = p' \circ q'$, $p, q \in X$.
(vi) $p \circ q' = p \circ q$, $p, q \in X$.
(vii) X has the elements $0 \in X$ and $1 \in X$ such that $p \circ 1 = p$, $1 \circ p = 1$, $p \circ 0 = p$, $0 \circ p = 0$, and $0' = 1$.

Remark 1.2. From (i) and (iii) of Definition 1.1 it follows that

$$p^2 \circ (q \circ p) = [(p \circ p) \circ q] \circ p \qquad \text{if} \quad p, q \in X$$

In general the left Jordan groupoid is noncommutative and also nonassociative. For more on this see Katrnoška (1993). We denote the left Jordan groupoid of X by $(X, \circ, 0, 1, ')$.

Example 1.3. If $U(R)$, resp. $P(R)$, are the sets of all idempotents, resp. projectors, of the $*$ ring R with the identity, then $(U(R), \underset{i}{\circ}, 0, 1, ')$, resp. $(P(R), \underset{i}{\circ}, 0, 1, ')$ $(i = 1, 2)$ are the left Jordan groupoids. The operations $\underset{i}{\circ}$: $U(R) \times U(R) \to U(R)$, resp. $\underset{i}{\circ}$: $P(R) \times P(R) \to P(R)$ $(i = 1, 2)$ are defined by setting

$$p \underset{1}{\circ} \circ q = p - 2pq - 2qp + 4qpq, \qquad p, q \in U(R) \text{ [resp. } p, q \in P(R)]$$

$$p \underset{2}{\circ} \circ q = q - 2pq - 2qp + 4pqp, \qquad p, q \in U(R) \text{ [resp. } p, q \in P(R)]$$

and the orthocomplement p' of $p \in U(R)$ [resp. $p \in P(R)$] by $p' = 1 - p$.

2. THEOREM OF THE CHARACTERIZATION

Our main aim is to prove the following theorem.

Theorem 2.1. Let R be an associative $*$ ring with the identity of the characteristic $\neq 2$ and let $U(R)$, resp. $P(R)$, be the sets of all idempotents, resp. projectors, of the $*$ ring R. Then $(U(R), \leq, 0, 1, ')$, resp. $(P(R), \leq, 0, 1, ')$ are the commutative Boolean algebras iff $(U(R), \underset{1}{\circ}, 0, 1, ')$, resp. $(P(R), \underset{1}{\circ}, 0, 1, ')$ are associative left Jordan groupoids.

Proof. (a) Necessary condition. We suppose that, for example, $(U(R), \leq, 0, 1, ')$ is a commutative Boolean algebra. If $p, q \in U(R)$, and also $pq = qp$, then

$$p \underset{1}{\circ} q = p$$

and we obtain

$$(p \underset{1}{\circ} q) \underset{1}{\circ} r = p \underset{1}{\circ} r = p = p \underset{1}{\circ} (q \underset{1}{\circ} r), \qquad p, q, r \in U(R)$$

The groupoid $(U(R), \underset{1}{\circ}, 0, 1, ')$ is also associative.

(b) The condition is sufficient. We show that if

$$(p \underset{1}{\circ} q) \underset{1}{\circ} q = p \underset{1}{\circ} (q \underset{1}{\circ} q) \qquad \text{for} \quad p, q \in U(R)$$

then $pq = qp$.

By (i) of Definition 1.1

$$(p \underset{1}{\circ} q) \underset{1}{\circ} q = p \underset{1}{\circ} q$$

Therefore we have

$$(p - 2pq - 2qp + 4qpq) \underset{1}{\circ} q = p - 2pq - 2qp + 4qpq$$

From the last equation it follows that

$$\begin{aligned} p &- 2pq - 2qp + 4qpq - 2pq + 4pq + 4qpq - 8qpp \\ &- 2qp + 4qpq + 4qp - 8qpq + 4qpq - 8qpq - 8qpq + 16qpq \\ &= p - 2pq - 2qp + 4qpq \end{aligned}$$

Then

$$2pq + 2qp - 4qpq = 0 \tag{1}$$

The multiplication of equation (1) on the right side and then on the left side by q gives

$$2pq + 2qpq - 4qpq = 0, \qquad 2qpq + 2qp - 4qpq = 0 \tag{2}$$

Further computations yield

$$2pq = 2qp$$

But the characteristic of R is different from 2. Therefore it follows that $pq = qp$, $p, q \in U(R)$, and $p \underset{1}{\circ} q = p$. If $p, q, r \in U(R)$, then we have

$$pq = qp, \qquad pr = rp, \qquad rq = qr$$

It must also necessary hold that

$$(p \underset{1}{\circ} q) \underset{1}{\circ} r = p \underset{1}{\circ} r = p = p \underset{1}{\circ} (q \underset{1}{\circ} r)$$

and the groupoid $(U(R), \underset{1}{\circ}, 0, 1, ')$ is also associative. QED

I want to emphasize that Theorem 2.1 gives the characterization of those Boolean algebras that have commuting elements [i.e., if $p, q \in U(R)$, then $pq = qp$].

3. SOME CONSEQUENCES

Finally we show the validity of a proposition concerning the orthogonal and compatible elements of $(U(R), \leq, 0, 1, ')$, resp. $(P(R), \leq, 0, 1, ')$.

Proposition 3.1. Let R be an associative $*$ ring with the identity, and let $U(R)$, resp. $P(R)$, be the set of all idempotents, resp. projectors, of the

$*$ ring R. If for $p, q \in U(R)$, resp. $p, q \in P(R)$, we have $p \leftrightarrow q$, then $p \underset{1}{\circ} q = p$, resp. $p \underset{2}{\circ} q = p$, and conversely.

Proof. Let $p, q \in U(R)$; then $p \leftrightarrow q$ implies $pq = qp$ and we have $p \underset{1}{\circ} q = p - 2pq - 2qp + 4qpq = p$ and conversely. QED

Remark 3.2. If for $p, q \in U(R)$, resp. $p, q \in P(R)$, $p \perp q$, then $p \leftrightarrow q$. Also, when $p \perp q$, then, according to Proposition 3.1, $p \underset{1}{\circ} q = p$.

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